A Bloore folyam alkalmazása az egy irányból koptatott szikla alakjának meghatározására

Sipos András

joint work with Máté Szondi

June 4, 2026

Outline

Background
- Motivation
- The Bloore flow

Results in 2D
- Model development
- Soliton solutions
- Comparison to experiments

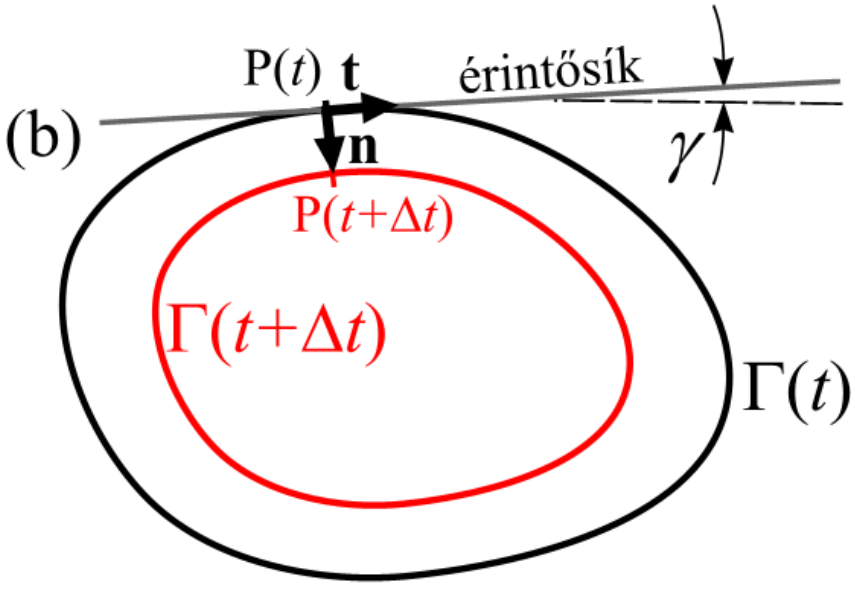
Results in 3D
- Model development
- Soliton solutions

Numerics
- Spectral method
- Chebfun solution

Motivation



A KOPÁS GEOMETRIAI MODELLEZÉSE

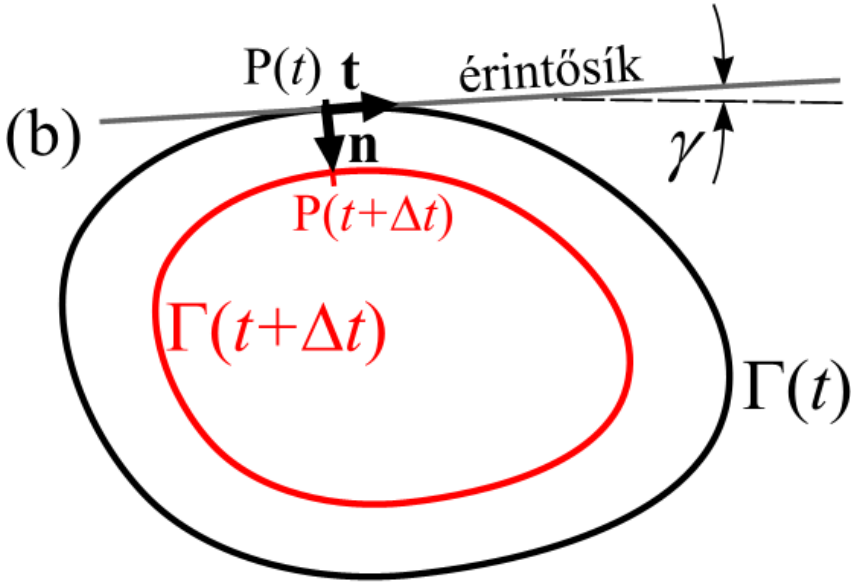


Legyen $\mathbf{r}(s,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ egy, időben változó síkgörbe pontjaiba mutató helyvektor! Sima, önátmetszés nélküli görbék alakfejlődése esetén egyértelmű megfeleltetést hozunk létre görbék egy családjának elemei között.
Az $\mathbf{n}$ normális-irányú megfeletetés esetén

$$\mathbf{r}_{,t} = f(\mathbf{r}, \mathbf{r}_{,s}, \mathbf{r}_{,ss})\mathbf{n} \tag{1}$$

ahol $f(.)$ a kopási sebességet meghatározó *kopási törvény*.

A KOPÁS GEOMETRIAI MODELLEZÉSE



Fizikai megfontolások alapján a kopási törvény elsősorban a kopó test *görbületének* függvénye, azaz síkban: $f(\kappa)$ és térben $f(H,K)$, ahol κ a görbület, H és K a felület Minkowski és Gauss görbületei.Például a

$$f(\kappa) = 1 + q\kappa \tag{2}$$

az ún. *Bloore modellnek* felel meg ($q > 0$).

A BLOORE EGYENLET ALAKJAI

Rögzített paraméterezésű Γ görbe esetén:

$$\mathbf{r}_{,t} = f(\kappa)\mathbf{n} = (1 + q\kappa)\mathbf{n} \tag{3}$$

Konvex test esetén a Γ görbét a γ érintőirány szerint is paraméterezhetjük. Ekkor:

$$\mathbf{r}_{,t} = f(\kappa)\mathbf{n} - f_{,\kappa}\kappa_{,\gamma}\mathbf{t} = (1 + q\kappa)\mathbf{n} - q\kappa_{,\gamma}\mathbf{t} \tag{4}$$

Megmutatható, az előző egyenlet a

$$h_{,t} = -f(\kappa) = -1 - q\kappa \tag{5}$$

egyenletre vezet, ahol $h(\gamma, t) = -\mathbf{r}(\gamma, t)\mathbf{n}$ a görbe *támasz-függvénye*. Végül, az is levezethető, hogy a Γ görbületfüggvénye a következő, PDE-nek tesz eleget:

$$\kappa_{,t} = \kappa^2 \left(f + f_\kappa \kappa_{\gamma\gamma} + f_{\kappa\kappa}\kappa_\gamma^2\right) = \kappa^2(1 + q\kappa + q\kappa_{,\gamma\gamma}). \tag{6}$$

EGY KIS KITÉRŐ:

A

$$\mathbf{r}_{,t} = (1 + q\kappa)\mathbf{n} - q\kappa_{,\gamma}\mathbf{t} \tag{7}$$

alakon jól látszik, hogy a görbület szélsőértékeinél a görbe normális irányba mozdul el, azaz az érintőirány nem változik.
A

$$\kappa_{,t} = \kappa^2(1 + q\kappa + q\kappa_{,\gamma\gamma}). \tag{8}$$

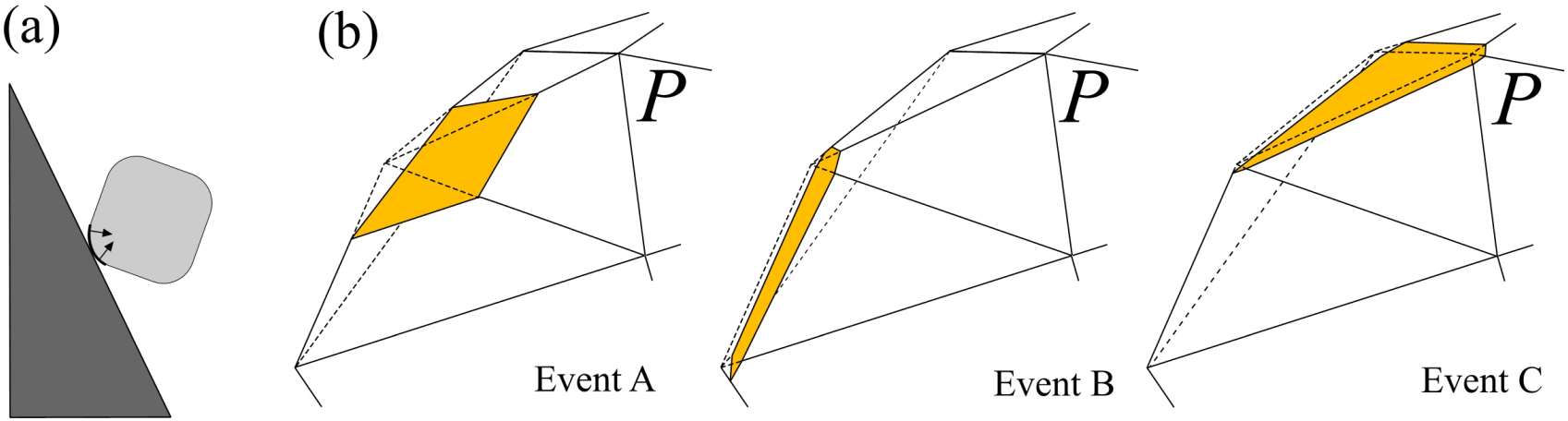
egyenlet γ szerinti deriválásával megmutatható, hogy a görbületi szélsőérték helyben marad, ha $\kappa_{,\gamma} = 0$ helyen a $\kappa_{,\gamma\gamma\gamma} = 0$ is teljesül.
Ehhez szorosan kapcsolódik, hogy a Bloore egyenlet a test szimmetriáját megőrzi (Bloore, 1977).

BLOORE FLOW IN 3D

Bloore equation in 3D:

$$\mathbf{r}_{,t} = (1 + bH + cK)\mathbf{n} = (cK + bH + 1)\mathbf{n} \tag{9}$$

intuitive picture:



The three terms in the geometric PDE are associated with three different abrader geometries:

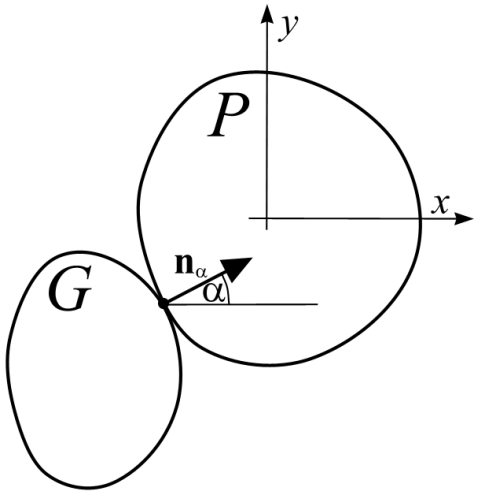
- A: plane
- B: needles
- C: dust

BLOORE FLOW IN 3D

Bloore equation in 3D:

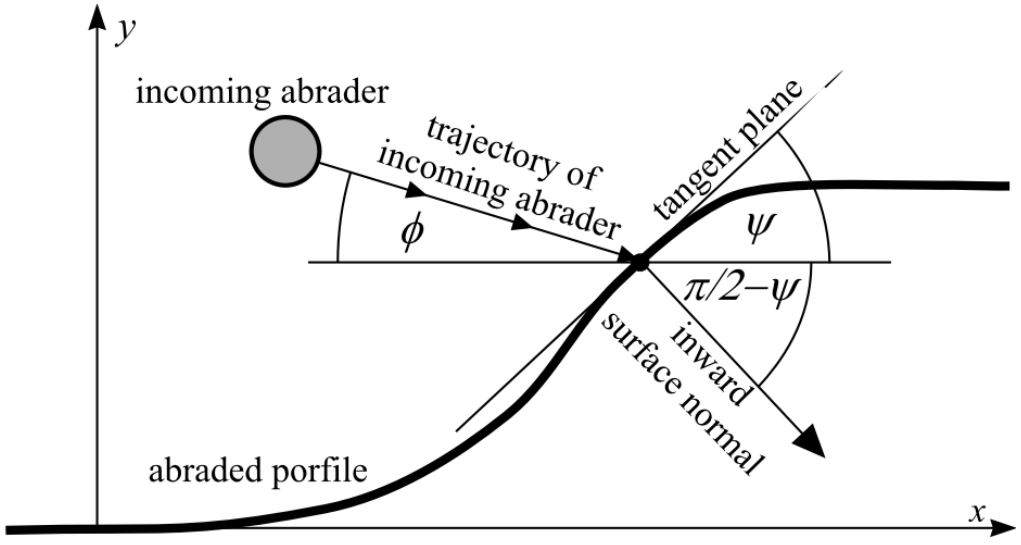
$$\mathbf{r}_{,t} = (1 + bH + cK)\mathbf{n}_\alpha \tag{10}$$

Note that $b > 0$ and/or $c > 0$ grab the case of finite abraders.



- general case: arrival direction of the abrader G is uniformly distributed, hence (10) holds at any surface point of P
- unidirectional case: one, fixed direction of hits, open curve.

UNIDIRECTIONAL BLOORE FLOW IN 2D [1]

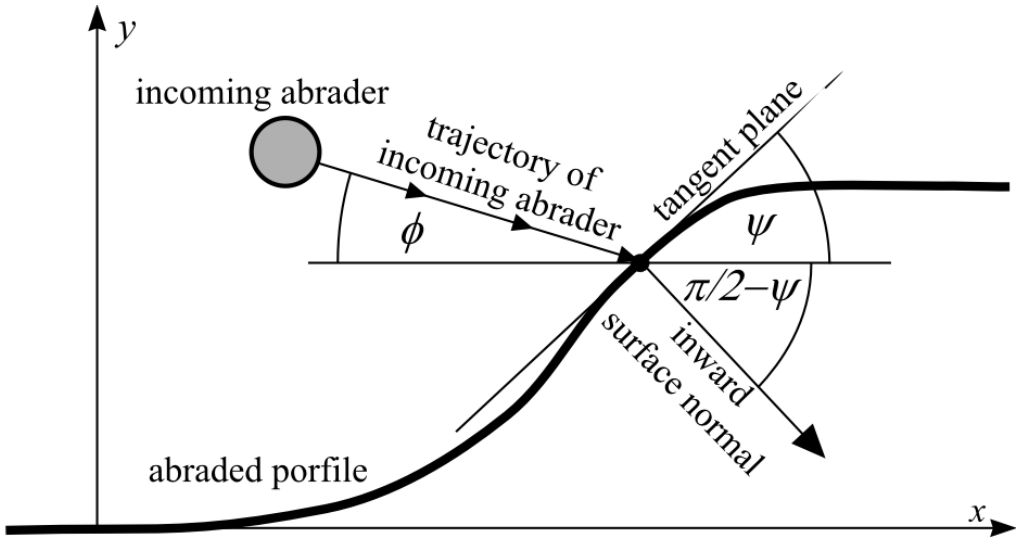


Let the profile be parametrized by y, hence the unknown is $x(y,t)$. Now, the magnitude of the normal speed can be expressed via

$$|\mathbf{v}| = \dot{x}\sin\psi = (1+b\kappa)\sin(\psi-\phi),$$
$$\dot{x} = (1+b\kappa)(\cos\phi - x'\sin\phi). \qquad (11)$$

[1] G. Domokos, G.W. Gibbons, A.A. Sipos: Circular, Stationary Profiles Emerging in Unidirectional Abrasion, *Math. Geosci*, **46**:483-491, 2014.

UNIDIRECTIONAL BLOORE FLOW IN 2D [1]



Without loss of generality let $\phi = 0$ that further simplifies to:

$$\dot{x} = (1 + b\kappa) = \left(1 + b\frac{x''}{(1 + x'^2)^{3/2}}\right). \tag{12}$$

[1] G. Domokos, G.W. Gibbons, A.A. Sipos: Circular, Stationary Profiles Emerging in Unidirectional Abrasion, *Math. Geosci*, **46**:483-491, 2014.

PLANAR MODEL - EXISTENCE OF TRAVELING WAVES

To prove soliton solutions the $y(x,t)$ parametrization is more convenient. Now

$$-\dot{y} = y'(1 - b\kappa) = y'\left(1 + b\frac{y''}{(1+y'^2)^{3/2}}\right). \tag{13}$$

Using the traveling-wave ansatz $y = f(x - ct)$ an ODE for $f(.)$ is obtained with solution

$$f(x-ct)=\frac{(b+cq-q+c(x-ct)-x+ct)(b-cq+q-c(x-ct)+x-ct)}{\sqrt{-(x-ct)^2c^2+2c(x-ct)^2-2(x-ct)c^2q+4c(x-ct)q-(x-ct)^2-2(x-ct)q-q^2c^2+2q^2c-q^2+b^2}(c-1)}$$

[1] G. Domokos, G.W. Gibbons, A.A. Sipos: Circular, Stationary Profiles Emerging in Unidirectional Abrasion, *Math. Geosci*, **46**:483-491, 2014.

PLANAR MODEL - EXISTENCE OF TRAVELING WAVES

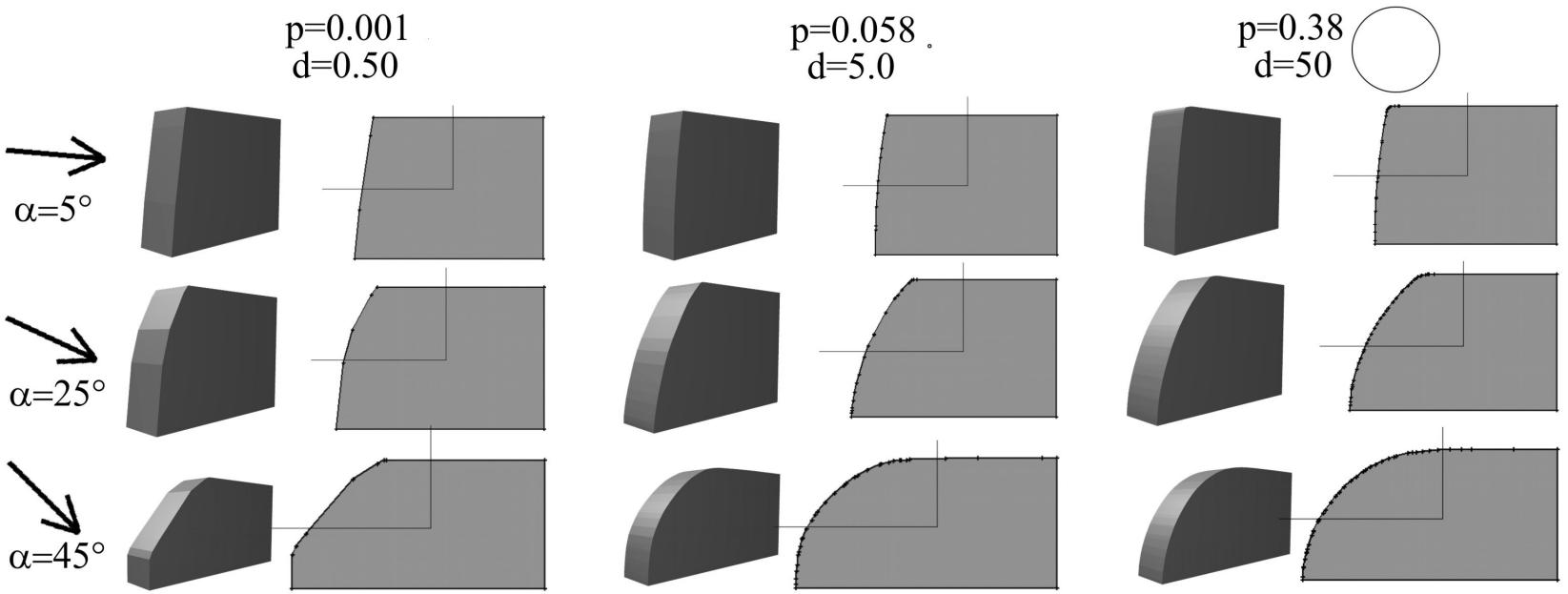
In fact, this is either a straight line or a circle with radius R. It moves to the right with speed

$$c = 1 + \frac{b}{R}. \tag{14}$$

Linear stability of traveling waves can be established analytically in [1],
Non-linear stability (attractivity of the traveling waves) was investigated numerically from real, scanned profiles.[1]
Question What determines the value of R?

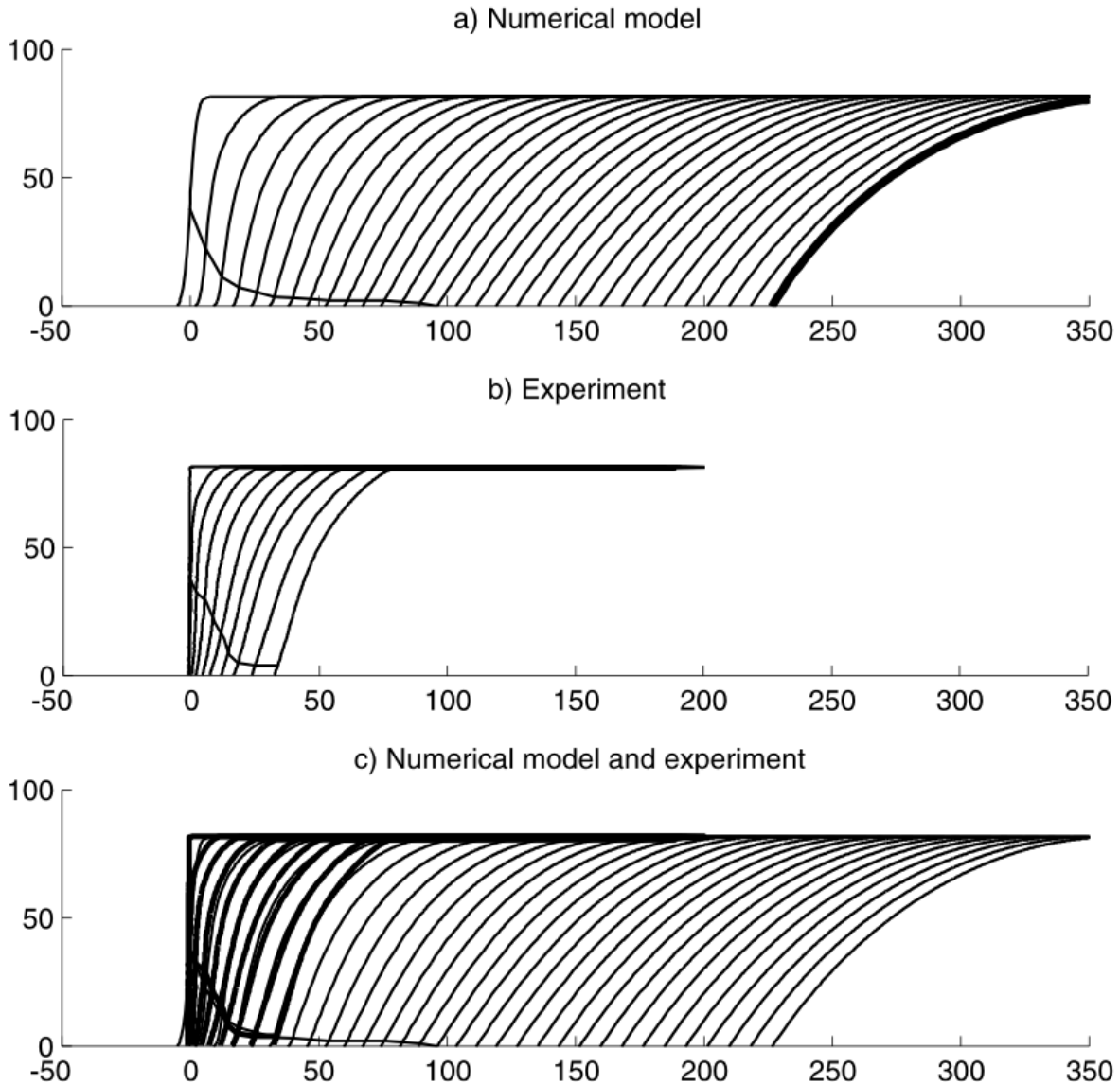
[1] G. Domokos, G.W. Gibbons, A.A. Sipos: Circular, Stationary Profiles Emerging in Unidirectional Abrasion, *Math. Geosci*, **46**:483-491, 2014.

PLANAR MODEL - NONLINEAR STABILITY



[2] Sipos A. Á., Domokos G., A. Wilson, N. Hovius: A Discrete Random Model Describing Bedrock Profile Abrasion, *Math. Geosci*, **43**:583-591, 2011.

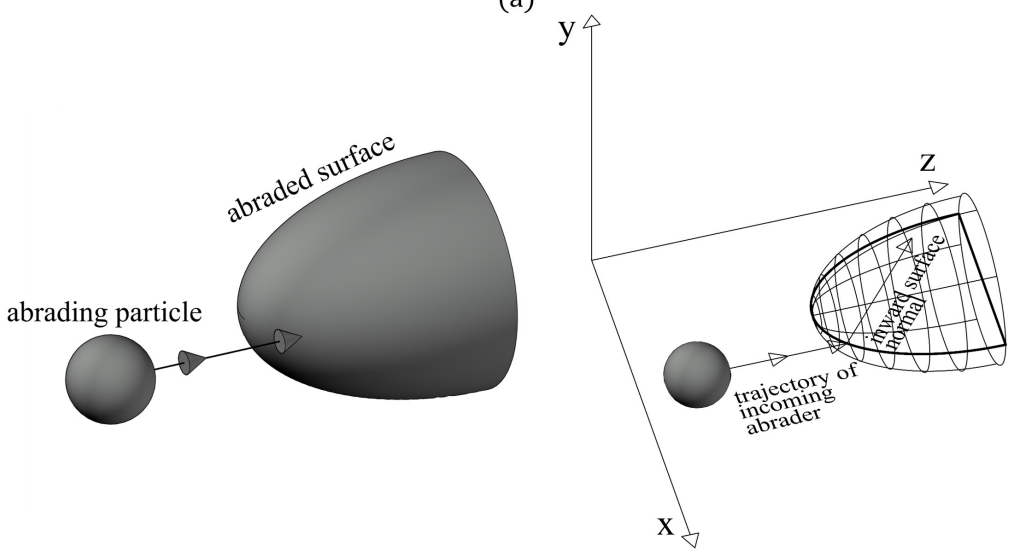
PLANAR MODEL - NONLINEAR STABILITY



UNIDIRECTIONAL BLOORE IN 3D



(a)



(b)

UNIDIRECTIONAL BLOORE IN 3D

Let the direction of abraders aligned with the positive z axis and the series of profiles parametrized via $z(x, y, t)$. Now ψ denotes the angle between normal of the surface and the z direction. Similarly to the 2D case we obtain

$$|\mathbf{v}| = z_{,t} \cos\psi = (1 + bH + cK)\cos(\psi) \tag{15}$$

$$\boxed{z_{,t} = (1 + bH + cK)}, \tag{16}$$

where the mean and Gaussian curvatures are given by

$$H := \frac{1}{2}\frac{(1 + z_y^2)z_{xx} + (1 + z_x^2)z_{yy} - 2z_x z_y z_{xy}}{(1 + z_x^2 + z_y^2)^{3/2}}, \tag{17}$$

$$K := \frac{z_{xx}z_{yy} - z_{xy}^2}{(1 + z_x^2 + z_y^2)^2}. \tag{18}$$

UNIDIRECTIONAL BLOORE IN 3D

The *traveling wave ansatz*, i.e. $z(x, y, t) = ut + f(x, y)$ yields

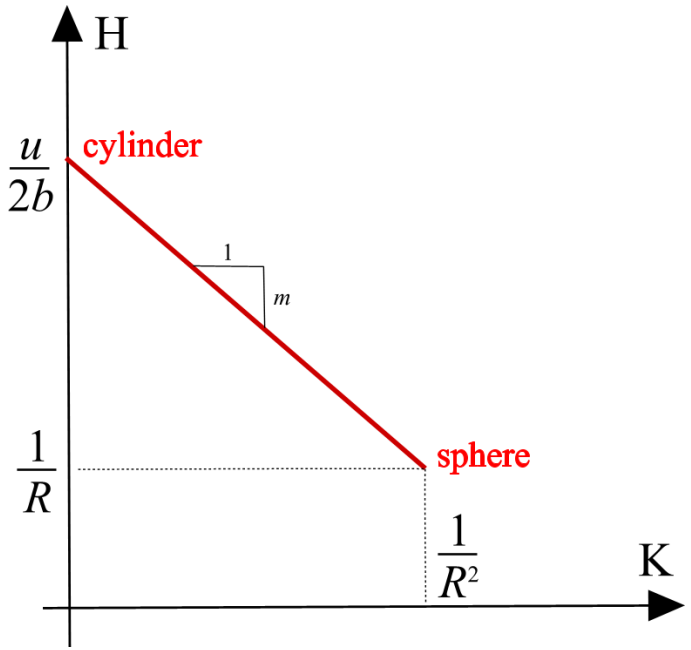
$$u = 1 + bH + cK, \tag{19}$$

which shows, that a traveling wave solution, and in specific $f(x, y)$ must be a *linear Weingarten surface*. Physical intuition tells $b > 0$ and $c > 0$, which means that in the $K - H$ space we expect a line with inclination m

$$m = -\frac{c}{2b}. \tag{20}$$

This can be viewed as a *generalized minimal surface problem*, if we prescribe Direchlet boundary for $f(x, y)$.
(Minimal surface problem: $H = 0$ with $f|_{\partial\Omega} = g$).

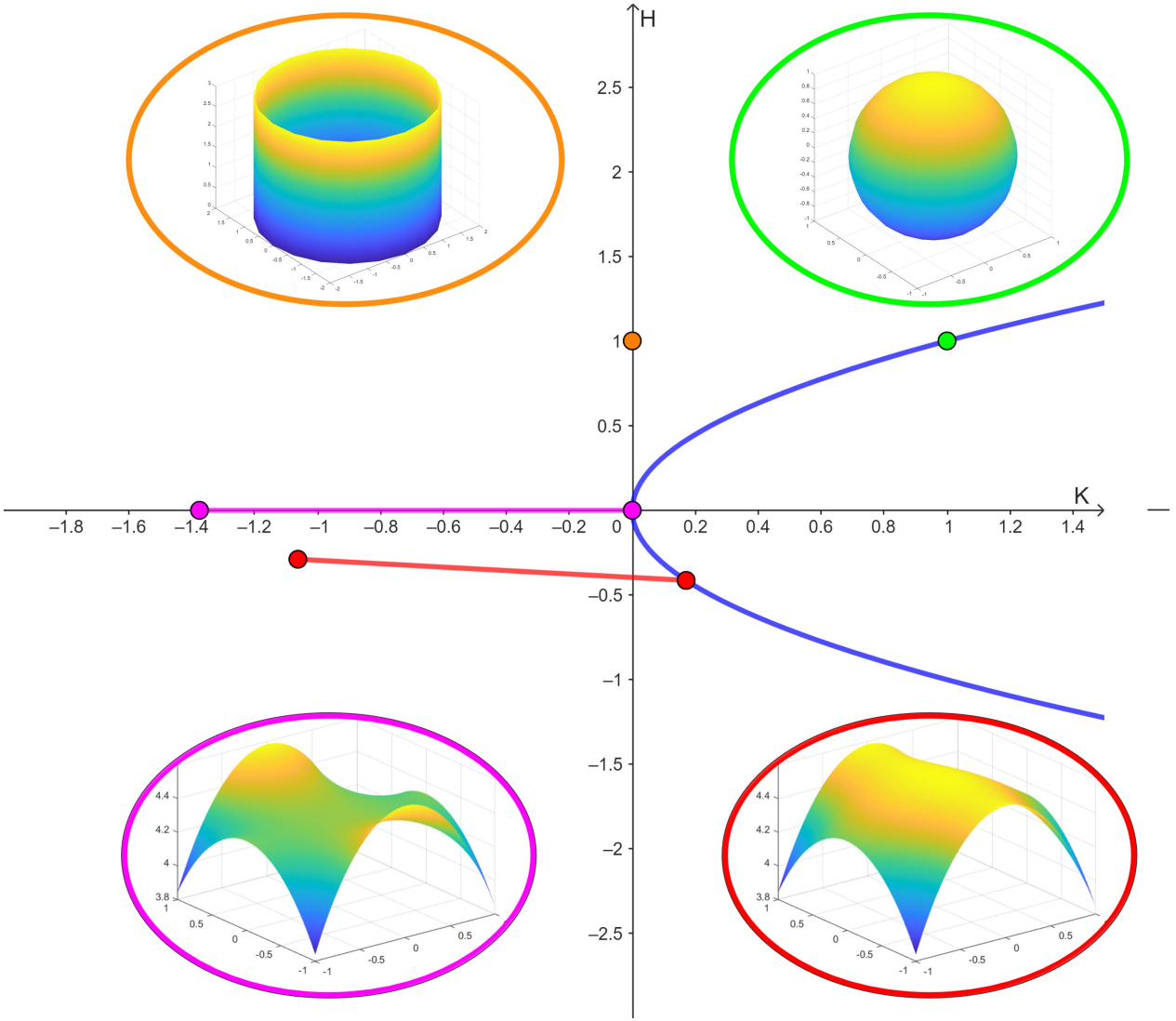
UNIDIRECTIONAL BLOORE IN 3D



It's not difficult to show, that the radius R of the sperical solution is

$$R = \frac{b + \sqrt{(b^2 + uc)}}{u}. \tag{21}$$

MORE ON THE $K - H$ SPACE



NUMERICAL SIMULATION

We need to

1. have an arbitrary (convex) profile bordered by an arbitrary (closed, non-intersecting) curve,
2. accurately approximate H and K in high number of spatial points,
3. run it for a very long time without accumulating singularities.

Level-set method is classical, but fails to meet points 1 and 3. Instead of that: a mesh-free (collocation) method following the lead of [3].

[3] K. Böhmer R. Schaback, A meshfree method for solving the Monge-Ampére equation, *Numerical Algorithms*, in press, 2018

SPECTRAL METHOD - 2D

Let M be a fixed positive integer. We seek $z(x,t)$ in the following form (the Fourier-series up to order M)

$$\hat{z}(x,t) = a_0(t) + \sum_{m=1}^{M} a_m(t)\cos(mx) + \sum_{m=1}^{M} b_m(t)\sin(mx). \tag{22}$$

Obviously

$$\hat{z}_x(x,t) = -\sum_{m=1}^{M} a_m(t)m\sin(mx) + \sum_{m=1}^{M} b_m(t)m\cos(mx), \tag{23}$$

$$\hat{z}_{xx}(x,t) = -\sum_{m=1}^{M} a_m(t)m^2\cos(mx) - \sum_{m=1}^{M} b_m(t)m^2\sin(mx) \tag{24}$$

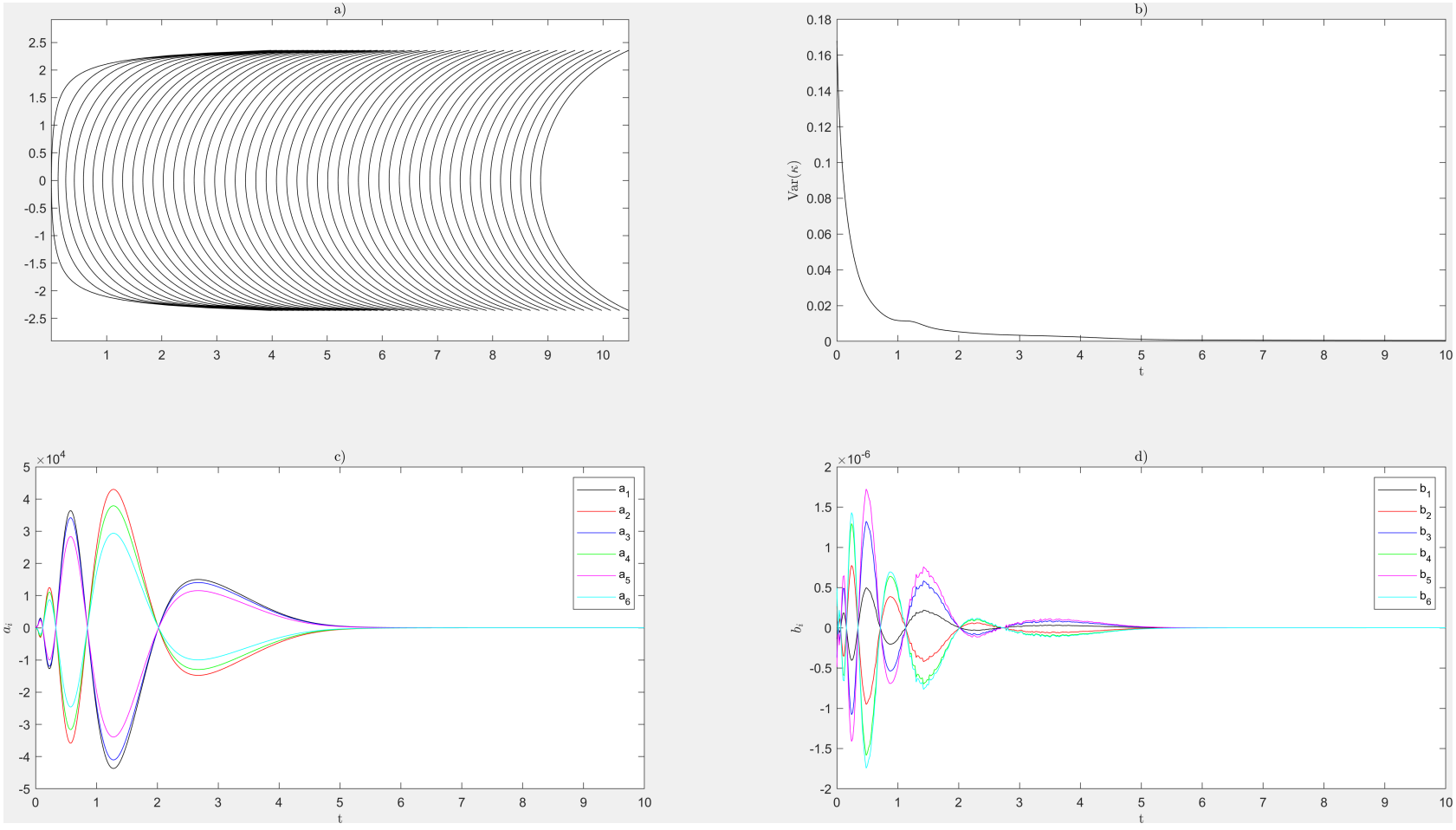
SPECTRAL METHOD - 2D

1. fix M, Δt and a set of sample locations x. Obtain $\hat{z}(x, 0)$.
2. obtain $\{a_i(0)\}_{i=0}^{M}$, $\{b_i(0)\}_{i=1}^{M}$ (note: x is fixed, this is simply a linear equation!)
3. compute $\hat{z}_x$ and $\hat{z}_{xx}$ and then $\kappa(x, t)$
4. produce the fourier-series of the speed:

$$1 + b \max(\kappa(x, t), 0) = c_0 + \sum_{m=1}^{M} c_m \cos(mx) + \sum_{m=1}^{M} d_m \sin(mx). \tag{25}$$

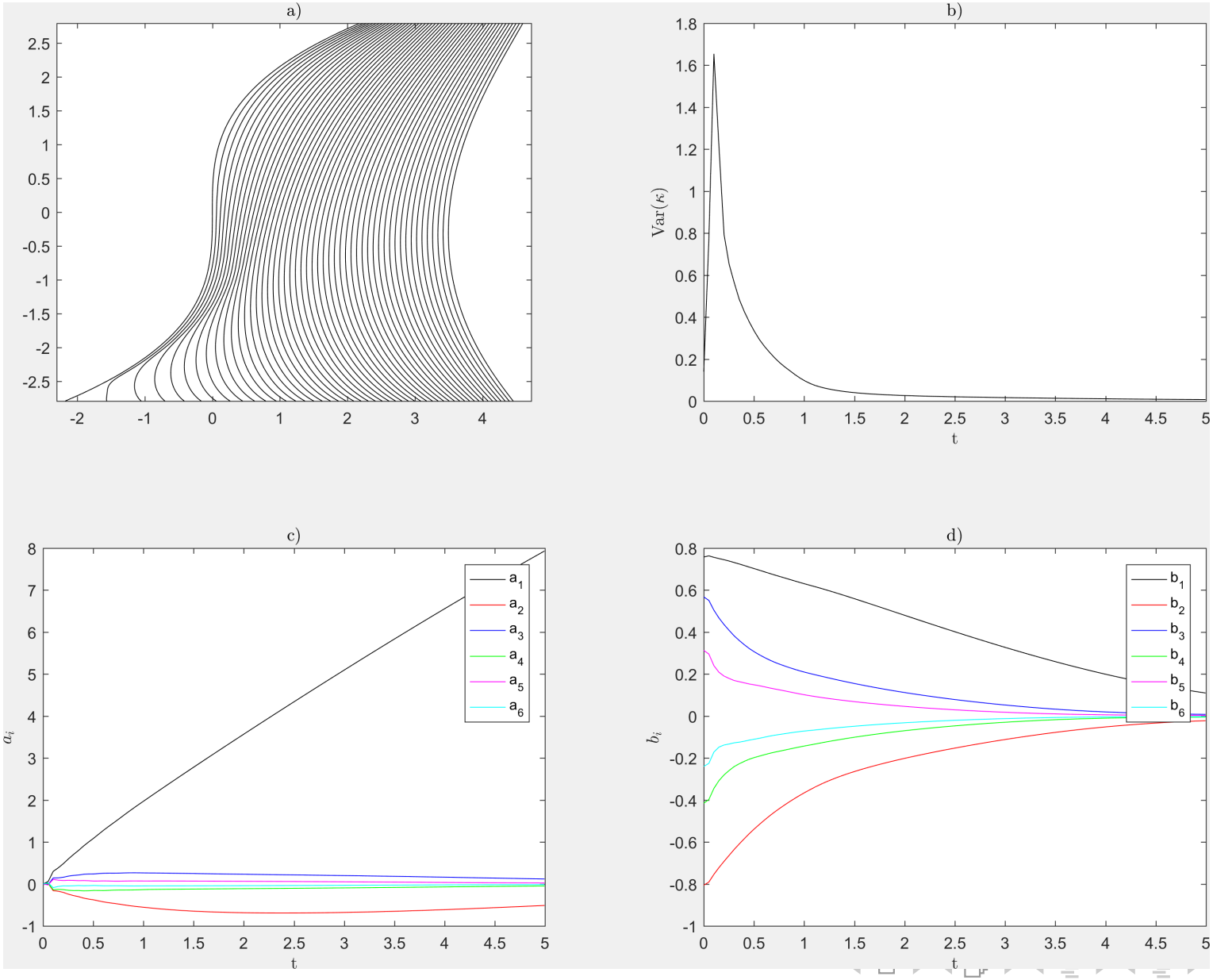
5. from $\dot{z} = 1 + b \max(\kappa(x), 0)$ it follows $\dot{a}_i = c_i$ and $\dot{b}_i = d_i \; \forall i$
6. goto 3.

2D - RESULTS



b=2, M=25, T=10, 5000 steps, scaled $\tanh(x)$ initial profile

2D - RESULTS



3D - RESULTS

Traveling boundaries (modeling abrasion):

$$\begin{cases} z_{,t} & = 1 + bH(z) + cK(z) & z(t,x,y) \in \mathbb{R}^+ \times \Omega \\ z(0,x,y) & = z_0(x,y) & z_0(x,y) \in \Omega \\ z(t,x,y)|_{\partial\Omega} & = ut + G(x,y)|_{\partial\Omega} & \forall t \end{cases} \tag{26}$$

Equivalent BVP (form finding):

$$\begin{cases} z_{,t} & = (1-u) + bH(z) + cK(z) & z(t,x,y) \in \mathbb{R}^+ \times \Omega \\ z(0,x,y) & = z_0(x,y) & z_0(x,y) \in \Omega \\ z(t,x,y)|_{\partial\Omega} & = G(x,y)|_{\partial\Omega} & \forall t \end{cases} \tag{27}$$

3D - RESULTS

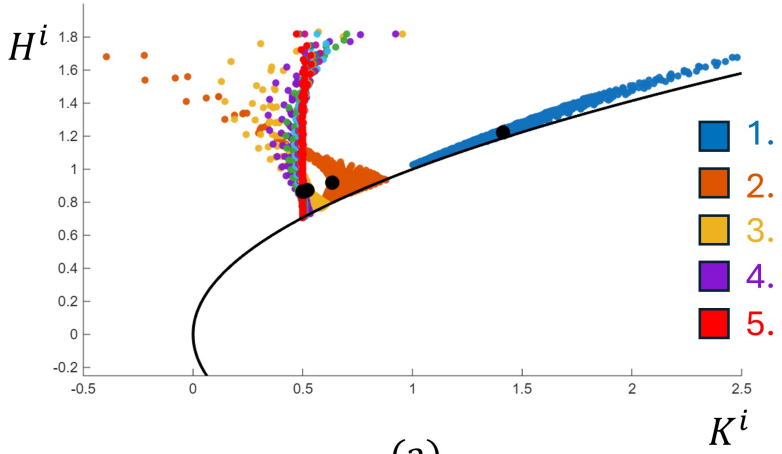
Half-implicit discrete scheme at a sufficiently small δt time step:

$$\frac{z^{t+1} - z^t}{\delta t} - \frac{1}{2} b \Delta z^{t+1} = F(z^t, \nabla z^t, \nabla^2 z^t) - \frac{1}{2} b \Delta z^t. \tag{28}$$

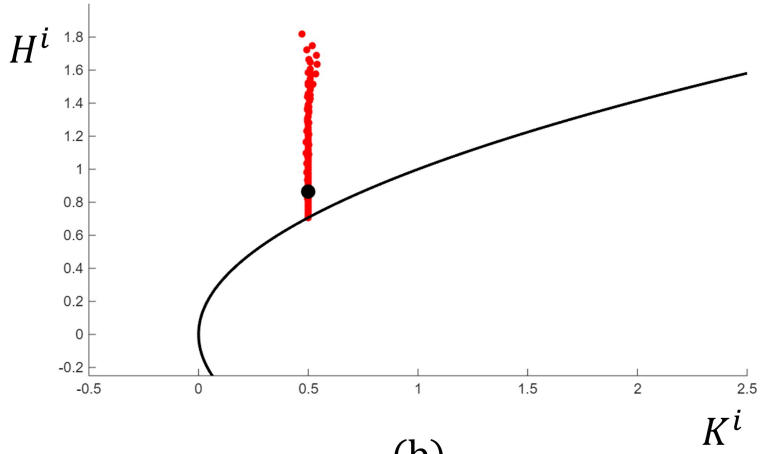
We us a custom made Matlab code (The MathWorks Inc., 2021) that uses the PDE solver of the Chebfun Toolbox (Driscoll et al., 2014) in order to establish the scheme above.

3D - RESULTS

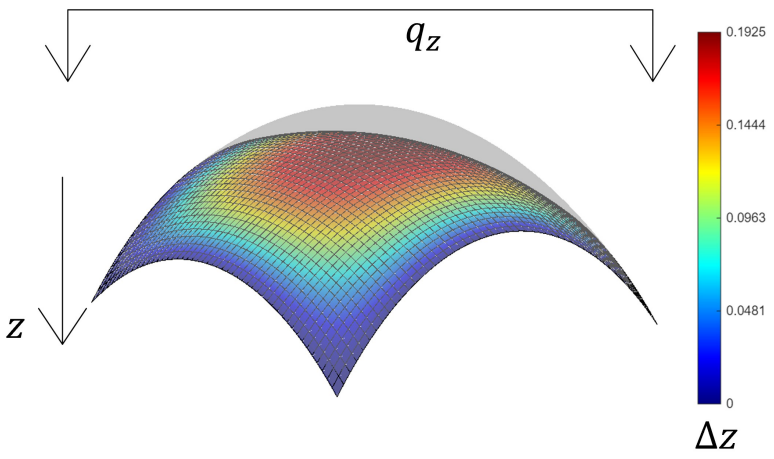
First results:



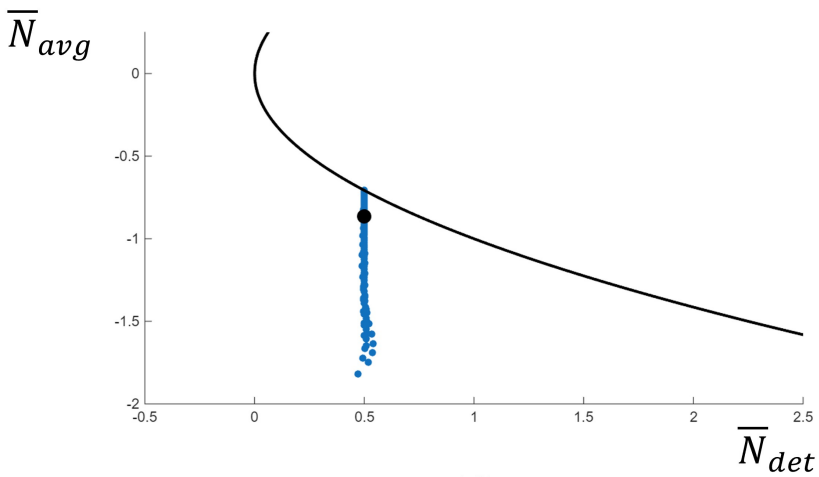
(a)



(b)



(c)



(d)

OTKA

?

Can the concept of equal energy snapshots developed for purely decaying dynamical systems predict the duration of transient behavior of mean curvature and Bloore flows, as PDE models of abrasional processes?

Can we infer from the shape of rocks the details of past abrasion events?

SUMMARY

Köszönöm a figyelmet!